

3-Lie Bialgebras ^{*}

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Abstract 3-Lie algebras have close relationships with many important fields in mathematics and mathematical physics. The paper concerns 3-Lie algebras. The concepts of 3-Lie coalgebras and 3-Lie bialgebras are given. The structures of such categories of algebras, and the relationships with 3-Lie algebras are studied. And the classification of 3-dimensional 3-Lie coalgebras and 3-Lie bialgebras over an algebraically closed field of characteristic zero are provided.

Key words: 3-Lie algebra; 3-Lie coalgebra; 3-Lie bialgebra

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1. Introduction

The notion of n -Lie algebra was introduced by Filippov in 1985 (cf [1]). n -Lie algebras are a kind of multiple algebraic systems appearing in many fields in mathematics and mathematical physics. Specially, the structure of 3-Lie algebras is applied to the study of the supersymmetry and gauge symmetry transformations of the world-volume theory of multiple coincident M2-branes; the Eq. (2.2) for a 3-Lie algebra is essential to define the action with $N = 8$ supersymmetry, and it can be regarded as a generalized Plucker relation in the physics literature, and so on (cf. [2, 3, 4, 5, 6]). Since the n -ary ($n \geq 3$) multiplication, the structure of n -Lie algebras (cf [7, 8, 9, 10, 11, 12, 13, 14], etc.) is more complicated than that of Lie algebras. We need to excavate more constructions of n -Lie algebras, and more relationships with algebraic systems related to n -Lie algebras, such as Lie algebras. Note that concepts of Lie coalgebras and Lie bialgebras (cf [15, 16, 17]) are important concepts in Lie algebras. For example, the coboundary Lie bialgebra associates to a solution of the Classical Yang-Baxter Equation, and it has been playing an important role in mathematics and physics. Motivated by this, we define the 3-Lie coalgebra and the 3-Lie bialgebra, and study the structure of them and the relationships with 3-Lie algebras. We also classify the 3-dimensional 3-coalgebras and 3-Lie bialgebras.

Throughout this paper, all algebras are over a field F of characteristic zero. Any bracket which is not listed in the multiplication table of a 3-Lie algebra or a 3-Lie coalgebra is assumed to be zero.

2. 3-Lie coalgebras

A 3-Lie algebra (L, μ) is a vector space L endowed with a 3-ary linear skewsymmetric multiplication μ satisfying the Jacobi identity: for every $x, y, z, u, v \in L$,

$$\mu(u, v, \mu(x, y, z)) = \mu(x, y, \mu(u, v, z)) + \mu(y, z, \mu(u, v, x)) + \mu(z, x, \mu(u, v, y)).$$

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We also describe the 3-Lie algebras as follows. A 3-Lie algebra (L, μ) is a vector space L endowed with a 3-ary linear multiplication $\mu : L \otimes L \otimes L \rightarrow L$ satisfying

$$\mu(1 - \tau) = 0, \quad (2.1)$$

$$\mu(1 \otimes 1 \otimes \mu)(1 - \omega_1 - \omega_2 - \omega_3) = 0, \quad (2.2)$$

where $\tau : L \otimes L \otimes L \rightarrow L \otimes L \otimes L$, for $\forall x_1, x_2, x_3 \in L$,

$$\tau(x_1 \otimes x_2 \otimes x_3) = \text{sign}(\sigma)x_{\sigma(1)} \otimes x_{\sigma(2)} \otimes x_{\sigma(3)}, \sigma \in S_3,$$

$$1, \omega_i : L \otimes L \otimes L \otimes L \otimes L \rightarrow L \otimes L \otimes L \otimes L \otimes L, \quad 1 \leq i \leq 3,$$

$$1(x_1 \otimes x_2 \otimes x_3 \otimes x_4 \otimes x_5) = x_1 \otimes x_2 \otimes x_3 \otimes x_4 \otimes x_5,$$

$$\omega_1(x_1 \otimes x_2 \otimes x_3 \otimes x_4 \otimes x_5) = x_3 \otimes x_4 \otimes x_1 \otimes x_2 \otimes x_5, \quad (2.3)$$

$$\omega_2(x_1 \otimes x_2 \otimes x_3 \otimes x_4 \otimes x_5) = x_4 \otimes x_5 \otimes x_1 \otimes x_2 \otimes x_3, \quad (2.4)$$

$$\omega_3(x_1 \otimes x_2 \otimes x_3 \otimes x_4 \otimes x_5) = x_5 \otimes x_3 \otimes x_1 \otimes x_2 \otimes x_4. \quad (2.5)$$

Let (L, μ) be a 3-Lie algebra, L^* be the dual space of L , then we get the dual mapping $\mu^* : L^* \rightarrow L^* \otimes L^* \otimes L^*$ of μ , satisfying for every $x, y, z \in L$ and $\xi, \eta, \zeta \in L^*$,

$$\langle \mu^*(\xi), x \otimes y \otimes z \rangle = \langle \xi, \mu(x, y, z) \rangle, \quad (2.6)$$

$$\langle \xi \otimes \eta \otimes \zeta, x \otimes y \otimes z \rangle = \langle \xi, x \rangle \langle \eta, y \rangle \langle \zeta, z \rangle, \quad (2.7)$$

where \langle, \rangle is the natural nondegenerate symmetric bilinear form on the vector space $L \oplus L^*$ is defined by $\langle \xi, x \rangle = \xi(x)$, $\xi \in L^*, x \in L$.

Then by the definition of 3-Lie algebras, we have $\text{Im}(\mu^*) \subseteq L^* \wedge L^* \wedge L^*$, and

$$(1 - \omega_1 - \omega_2 - \omega_3)(1 \otimes 1 \otimes \mu^*)\mu^* = 0, \text{ that is, for every } x, y, z, u, v \in L \text{ and } \xi, \eta, \zeta, \alpha, \beta \in L^*,$$

$$\langle (1 - \omega_1 - \omega_2 - \omega_3)(1 \otimes 1 \otimes \mu^*)\mu^*(\xi), x \otimes y \otimes z \otimes u \otimes v \rangle$$

$$= \langle (1 \otimes 1 \otimes \mu^*)\mu^*(\xi), (1 - \omega_1 - \omega_2 - \omega_3)(x \otimes y \otimes z \otimes u \otimes v) \rangle$$

$$= \langle \mu^*(\xi), (1 \otimes 1 \otimes \mu)(1 - \omega_1 - \omega_2 - \omega_3)(x \otimes y \otimes z \otimes u \otimes v) \rangle$$

$$= \langle \xi, \mu(1 \otimes 1 \otimes \mu)(1 - \omega_1 - \omega_2 - \omega_3)(x \otimes y \otimes z \otimes u \otimes v) \rangle. \quad (2.8)$$

By the above discussion, we give the definition of 3-Lie coalgebra.

Definition 2.1 A 3-Lie coalgebra (L, Δ) is a vector space L with a linear mapping $\Delta : L \rightarrow L \otimes L \otimes L$ satisfying

$$(1) \text{Im}(\Delta) \subset L \wedge L \wedge L, \quad (2.9)$$

$$(2) (1 - \omega_1 - \omega_2 - \omega_3)(1 \otimes 1 \otimes \Delta)\Delta = 0, \quad (2.10)$$

where $\omega_1, \omega_2, \omega_3 : L^{\otimes 5} \rightarrow L^{\otimes 5}$ satisfying identities (2.3), (2.4) and (2.5), respectively, and 1 is the identity mapping of $L^{\otimes 5}$.

Now we study 3-Lie coalgebras by means of structural constants. Let (L, Δ) be a 3-Lie coalgebra with a basis e_1, \dots, e_m . Assume

$$\Delta(e_l) = \sum_{1 \leq r < s < t \leq m} c_{rst}^l e_r \wedge e_s \wedge e_t, \quad c_{rst}^k \in F, \quad 1 \leq l \leq m. \quad (2.11)$$

Then we have

$$(1 \otimes 1 \otimes \Delta) \circ \Delta e_l = \sum_{r < s < t} c_{rst}^l e_r \wedge e_s \wedge \Delta e_t = \sum_{r < s < t} \sum_{i < j < k} c_{ijk}^t c_{rst}^l e_r \wedge e_s \wedge e_i \wedge e_j \wedge e_k, \quad (2.12)$$

$$\begin{aligned} & (1 - \omega_1 - \omega_2 - \omega_3) \circ (1 \otimes 1 \otimes \Delta) \circ \Delta e_l \\ &= (1 - \omega_1 - \omega_2 - \omega_3) \left(\sum_{r < s < t} \sum_{i < j < k} c_{ijk}^t c_{rst}^l e_r \wedge e_s \wedge e_i \wedge e_j \wedge e_k \right) \\ &= \sum_{r < s < t} \sum_{i < j < k} c_{ijk}^t c_{rst}^l e_r \wedge e_s \wedge e_i \wedge e_j \wedge e_k - \sum_{r < s < t} \sum_{i < j < k} c_{rsk}^t c_{ijt}^l e_i \wedge e_j \wedge e_r \wedge e_s \wedge e_k \\ &\quad - \sum_{r < s < t} \sum_{i < j < k} c_{jkt}^l c_{rsi}^t e_j \wedge e_k \wedge e_r \wedge e_s \wedge e_i - \sum_{r < s < t} \sum_{i < j < k} c_{kit}^l c_{rsj}^t e_k \wedge e_i \wedge e_r \wedge e_s \wedge e_j \\ &= \sum_{r < s < t} \sum_{i < j < k} [c_{ijk}^t c_{rst}^l - c_{rsk}^t c_{ijt}^l - c_{jkt}^l c_{rsi}^t - c_{kit}^l c_{rsj}^t] e_r \wedge e_s \wedge e_i \wedge e_j \wedge e_k, \end{aligned} \quad (2.13)$$

$$c_{i_1 i_2 i_3}^k = \text{sgn}(\sigma) c_{i_{\sigma(1)} i_{\sigma(2)} i_{\sigma(3)}}^k, \quad 1 \leq i_1, i_2, i_3 \leq m. \quad (2.14)$$

Therefore,

$$\sum_{k=1}^n (c_{ijk}^t c_{rst}^l - c_{rsk}^t c_{ijt}^l - c_{jkt}^l c_{rsi}^t - c_{kit}^l c_{rsj}^t) = 0, \quad 1 \leq i, j, k, l \leq m. \quad (2.15)$$

Following the above discussions, we obtain the structural description of 3-Lie coalgebras in terms of structural constants.

Theorem 2.2 Let L be an n -dimensional vector space with a basis e_1, \dots, e_m , $\Delta : L \rightarrow L \otimes L \otimes L$ be defined as (2.11). Then (L, Δ) is a 3-Lie coalgebra if and only if the constants c_{ijk}^l , $1 \leq i, j, k \leq m$ satisfy identities (2.14) and (2.15). \square

Now let (L, μ) be a 3-Lie algebra with a basis e_1, e_2, \dots, e_m , and the multiplication of L in the basis is as follows

$$\mu(e_i, e_j, e_k) = \sum_{l=1}^n c_{ijk}^l e_l, \quad c_{ijk}^l \in F, \quad 1 \leq i, j, k, l \leq m. \quad (2.16)$$

By the skew-symmetry of the multiplication of 3-Lie algebras, we have

$$c_{i_1 i_2 i_3}^l = \text{sgn}(\sigma) c_{i_{\sigma(1)} i_{\sigma(2)} i_{\sigma(3)}}^l,$$

that is, $c_{i_1 i_2 i_3}^l$, $1 \leq l, i_1, i_2, i_3 \leq m$ satisfy the identity (2.14). By the Jacobi identity (2.2)

$$\mu((e_i, e_j, e_k), e_s, e_t) = \mu(\mu(e_i, e_s, e_t), e_j, e_k) + \mu(e_i, \mu(e_j, e_s, e_t), e_k) + \mu(e_i, e_j, \mu(e_k, e_s, e_t)).$$

Since

$$\begin{aligned} \mu(\mu(e_i, e_j, e_k), e_s, e_t) &= \mu\left(\sum_{l=1}^n c_{ijk}^l e_l, e_s, e_t\right) = \sum_{l=1}^n c_{ijk}^l \mu(e_l, e_s, e_t) = \sum_{r=1}^n \sum_{l=1}^n c_{ijk}^l c_{lst}^r e_r, \\ \mu(\mu(e_i, e_s, e_t), e_j, e_k) &+ \mu(e_i, \mu(e_j, e_s, e_t), e_k) + \mu(e_i, e_j, \mu(e_k, e_s, e_t)) \\ &= \mu\left(\sum_{l=1}^n c_{ist}^l e_l, e_j, e_k\right) + \mu\left(e_i, \sum_{l=1}^n c_{jst}^l e_l, e_k\right) + \mu\left(e_i, e_j, \sum_{l=1}^n c_{kst}^l e_l\right) \\ &= \sum_{r=1}^n \sum_{l=1}^n c_{ist}^l c_{ljk}^r e_r + \sum_{r=1}^n \sum_{l=1}^n c_{jst}^l c_{ilk}^r e_r + \sum_{r=1}^n \sum_{l=1}^n c_{kst}^l c_{ijl}^r e_r, \end{aligned}$$

we have $\sum_{l=1}^n (c_{ijk}^l c_{lst}^r + c_{lkj}^r c_{ist}^l + c_{lik}^r c_{jst}^l + c_{lji}^r c_{kst}^l) = 0$, that is, $\{c_{i_1 i_2 i_3}^l, 1 \leq i_1, i_2, i_3 \leq n\}$ satisfies the identity (2.15).

Let L^* be the dual space of L , e^1, \dots, e^m be the dual basis of e_1, \dots, e_m , that is, $\langle e^i, e_j \rangle = \delta_{ij}$, $1 \leq i, j \leq m$. Assume $\mu^* : L^* \rightarrow L^* \otimes L^* \otimes L^*$ is the dual mapping of μ defined by (2.6), that is for every $\xi \in L^*, x, y, z \in L$, $\langle \mu^*(\xi), x \otimes y \otimes z \rangle = \langle \xi, \mu(x, y, z) \rangle$.

Then for every $1 \leq l \leq m$, we have

$$\mu^*(e^l) = \sum_{1 \leq i < j < k \leq n} c_{ijk}^l e^i \wedge e^j \wedge e^k. \quad (2.17)$$

Follows the identities (2.11) and (2.15), (L^*, μ^*) is a 3-Lie coalgebra.

Conversely, if (L, Δ) is a 3-Lie coalgebra with a basis e_1, \dots, e_m satisfying (2.11), L^* is the dual space of L with the dual basis e^1, \dots, e^m . Then the dual mapping $\Delta^* : L^* \otimes L^* \otimes L^* \rightarrow L^*$ of Δ satisfies, for every $\xi, \eta, \zeta \in L^*, x \in L$,

$$\langle \Delta^*(\xi, \eta, \zeta), x \rangle = \langle \xi \otimes \eta \otimes \zeta, \Delta(x) \rangle. \quad (2.18)$$

Then $\Delta^*(e^i, e^j, e^k) = \sum_{l=1}^n c_{ijk}^l e^l$, $c_{ijk}^l \in F$, $1 \leq i, j, k, l \leq m$ and Δ^* satisfies identity (2.15).

Summarizing above discussions, we have the following result.

Theorem 2.3 Let L be a vector space, $\Delta : L \rightarrow L \otimes L \otimes L$. Then (L, Δ) is a 3-Lie coalgebra if and only if (L^*, Δ^*) is a 3-Lie algebra. \square

We can also give an equivalence description of Theorem 2.3.

Theorem 2.4 Let L be a vector space over a field F , and $\mu : L \otimes L \otimes L \rightarrow L$ be a 3-ary linear mapping. Then (L, μ) is a 3-Lie algebra with the multiplication μ if and only if (L^*, μ^*) is a 3-Lie coalgebra with $\mu^* : L^* \rightarrow L^* \otimes L^* \otimes L^*$, where μ^* is the dual mapping of μ . \square

Example 2.5 Let L be a 5-dimensional 3-Lie algebra with the following multiplication

$$\mu(e_2, e_3, e_4) = e_1, \mu(e_3, e_4, e_5) = e_3 + 2e_2, \mu(e_2, e_4, e_5) = e_3, \mu(e_1, e_4, e_5) = e_1,$$

where e_1, e_2, e_3, e_4, e_5 is a basis of L . By Theorem 2.3, (L^*, μ^*) is a 3-Lie coalgebra with the linear mapping $\mu^* : L^* \rightarrow L^* \otimes L^* \otimes L^*$ satisfying

$$\mu^*(e^1) = e^2 \wedge e^3 \wedge e^4 + e^1 \wedge e^4 \wedge e^5, \mu^*(e^3) = e^3 \wedge e^4 \wedge e^5 + e^2 \wedge e^4 \wedge e^5, \mu^*(e^2) = \alpha e^3 \wedge e^4 \wedge e^5,$$

where $e^1, e^2, e^3, e^4, e^5 \in L^*$ is the dual basis of e_1, e_2, e_3, e_4, e_5 . \square

Example 2.6 Let (L, Δ) be a 4-dimensional 3-Lie coalgebra with a basis e_1, e_2, e_3, e_4 , and satisfying $\Delta(e_2) = \alpha e_3 \wedge e_1 \wedge e_4, \Delta(e_3) = e_3 \wedge e_1 \wedge e_4 + e_2 \wedge e_1 \wedge e_4$.

(L^*, Δ^*) is a 3-Lie algebra with the 3-ary linear skew-symmetric mapping $\Delta^* : L^* \otimes L^* \otimes L^* \rightarrow L^*$ satisfying

$$\Delta^*(e^2, e^3, e^4) = \alpha e^2 + e^3, \Delta^*(e^2, e^1, e^4) = e^3, \text{ where } e^1, e^2, e^3, e^4 \in L^* \text{ is the dual basis of } e_1, e_2, e_3, e_4. \quad \square$$

Definition 2.7 Let (L_1, Δ_1) and (L_2, Δ_2) be 3-Lie coalgebras. If there is a linear isomorphism $\varphi : L_1 \rightarrow L_2$ satisfying

$$(\varphi \otimes \varphi \otimes \varphi)(\Delta_1(e)) = \Delta_2(\varphi(e)), \text{ for every } e \in L_1, \quad (2.19)$$

then (L_1, Δ_1) is isomorphic to (L_2, Δ_2) , and φ is called a 3-Lie coalgebra isomorphism, where

$$(\varphi \otimes \varphi \otimes \varphi) \sum_i (a_i \otimes b_i \otimes c_i) = \sum_i \varphi(a_i) \otimes \varphi(b_i) \otimes \varphi(c_i). \quad (2.20)$$

Theorem 2.8 Let (L_1, Δ_1) and (L_2, Δ_2) be 3-Lie coalgebras. Then $\varphi : L_1 \rightarrow L_2$ is a 3-Lie coalgebra isomorphism from (L_1, Δ_1) to (L_2, Δ_2) if and only if the dual mapping $\varphi^* : L_2^* \rightarrow L_1^*$ is a 3-Lie algebra isomorphism from (L_2^*, Δ_2^*) to (L_1^*, Δ_1^*) , where for every $\xi \in L_2^*, v \in L_1, \langle \varphi^*(\xi), v \rangle = \langle \xi, \varphi(v) \rangle$.

Proof Since (L_1, Δ_1) and (L_2, Δ_2) are 3-Lie coalgebras, (L_1^*, Δ_1^*) and (L_2^*, Δ_2^*) are 3-Lie algebras. Let $\varphi : L_1 \rightarrow L_2$ be a 3-Lie coalgebra isomorphism from (L_1, Δ_1) to (L_2, Δ_2) . Then the dual mapping $\varphi^* : L_2^* \rightarrow L_1^*$ is a linear isomorphism. And for every $\xi, \eta, \zeta \in L_2^*, x \in L_1$, by identities (2.6) and (2.18)

$$\begin{aligned} \langle \varphi^* \Delta_2^*(\xi, \eta, \zeta), x \rangle &= \langle \Delta_2^*(\xi, \eta, \zeta), \varphi(x) \rangle \\ &= \langle \xi \otimes \eta \otimes \zeta, \Delta_2(\varphi(x)) \rangle = \langle \xi \otimes \eta \otimes \zeta, (\varphi \otimes \varphi \otimes \varphi) \Delta_1(x) \rangle \\ &= \langle \varphi^*(\xi) \otimes \varphi^*(\eta) \otimes \varphi^*(\zeta), \Delta_1(x) \rangle \\ &= \langle \Delta_1^*(\varphi^*(\xi), \varphi^*(\eta), \varphi^*(\zeta)), x \rangle. \end{aligned}$$

It follows $\varphi^* \Delta_2^*(\xi, \eta, \zeta) = \Delta_1^*(\varphi^*(\xi), \varphi^*(\eta), \varphi^*(\zeta))$, that is, $\varphi^* : L_2^* \rightarrow L_1^*$ is a 3-Lie algebra isomorphism from (L_2^*, Δ_2^*) to (L_1^*, Δ_1^*) .

Similarly, the conversion holds. \square

3. 3-Lie bialgebras

In this section we define 3-Lie bialgebra, which is an algebraic system with 3-ary multiplication structures of 3-Lie algebra and 3-Lie coalgebra, simultaneously. We first give the definition.

Definition 3.1 A 3-Lie bialgebra is a triple (L, μ, Δ) such that

- (1) (L, μ) is a 3-Lie algebra with the multiplication $\mu : L \wedge L \wedge L \rightarrow L$,
- (2) (L, Δ) is a 3-Lie coalgebra with $\Delta : L \rightarrow L \wedge L \wedge L$,
- (3) Δ and μ satisfy the following condition

$$\Delta \mu(x, y, z) = ad_\mu^{(3)}(x, y) \Delta(z) + ad_\mu^{(3)}(y, z) \Delta(x) + ad_\mu^{(3)}(z, x) \Delta(y), \quad (3.1)$$

where $ad_\mu(x, y) : L \wedge L \rightarrow End(L)$, $ad_\mu(x, y)(z) = \mu(x, y, z)$ for $x, y, z \in L$;

$ad_\mu^{(3)}(x, y), ad_\mu^{(3)}(z, x), ad_\mu^{(3)}(y, z) : L \otimes L \otimes L \rightarrow L \otimes L \otimes L$ are 3-ary linear mappings satisfying for every $u, v, w \in L$

$$\begin{aligned} ad_\mu^{(3)}(x, y)(u \otimes v \otimes w) &= (ad_\mu(x, y) \otimes 1 \otimes 1)(u \otimes v \otimes w) \\ &+ (1 \otimes ad_\mu(x, y) \otimes 1)(u \otimes v \otimes w) + (1 \otimes 1 \otimes ad_\mu(x, y))(u \otimes v \otimes w) \\ &= \mu(x, y, u) \otimes v \otimes w + u \otimes \mu(x, y, v) \otimes w + u \otimes v \otimes \mu(x, y, w), \end{aligned} \quad (3.2)$$

and the similar action for $ad_\mu^{(3)}(z, x)$ and $ad_\mu^{(3)}(y, z)$. \square

The condition (1) is equivalent to that (L^*, μ^*) is a 3-Lie coalgebra with $\mu^* : L^* \rightarrow L^* \wedge L^* \wedge L^*$ defined by (2.6).

The condition (2) is equivalent to that (L^*, Δ^*) is a 3-Lie algebra with the 3-Lie bracket $\Delta^* : L^* \wedge L^* \wedge L^* \rightarrow L^*$ defined by (2.18) satisfying for every permutation $\sigma \in S_3$, and $\xi_1, \dots, \xi_5 \in L^*$,

$$\begin{aligned} \Delta^*(\xi_1, \xi_2, \xi_3) &= \text{sign}(\sigma) \Delta^*(\xi_{\sigma(1)}, \xi_{\sigma(2)}, \xi_{\sigma(3)}), \\ \Delta^*(\xi_1, \xi_2, \Delta^*(\xi_3, \xi_4, \xi_5)) \\ &= \Delta^*(\xi_4, \xi_5, \Delta^*(\xi_1, \xi_2, \xi_3)) + \Delta^*(\xi_5, \xi_3, \Delta^*(\xi_1, \xi_2, \xi_4)) + \Delta^*(\xi_3, \xi_4, \Delta^*(\xi_1, \xi_2, \xi_5)). \end{aligned}$$

An alternate way of writing the condition (3) is for every $x, y, z \in L, \xi, \eta, \zeta \in L^*$,

$$\begin{aligned} \langle \Delta^*(\xi, \eta, \zeta), \mu(x, y, z) \rangle &= \langle \xi \otimes \eta \otimes \zeta, \Delta(\mu(x, y, z)) \rangle = \langle \xi \otimes \eta \otimes \zeta, \text{ad}_\mu^{(3)}(x, y) \Delta(z) \rangle \\ &+ \langle \xi \otimes \eta \otimes \zeta, \text{ad}_\mu^{(3)}(z, x) \Delta(y) \rangle + \langle \xi \otimes \eta \otimes \zeta, \text{ad}_\mu^{(3)}(y, z) \Delta(x) \rangle. \end{aligned}$$

Example 3.2 Let L be a 4-dimensional vector space with a basis e_1, e_2, e_3, e_4 . Defines

$$\begin{aligned} \mu : L \wedge L \wedge L &\rightarrow L, \quad \begin{cases} \mu(e_1, e_3, e_4) = e_1, \\ \mu(e_2, e_3, e_4) = e_2; \end{cases} \\ \Delta : L &\rightarrow L \wedge L \wedge L, \quad \begin{cases} \Delta(e_1) = e_3 \wedge e_2 \wedge e_4, \\ \Delta(e_3) = e_1 \wedge e_2 \wedge e_4. \end{cases} \end{aligned}$$

Then by the direct computation, the triple (L, μ, Δ) is a 3-Lie bialgebra. \square

Let (L, μ) be a 3-Lie algebra, for $\forall x, y, z \in L$, $\text{ad}_\mu : L \wedge L \rightarrow \text{gl}(L)$, $\text{ad}_\mu(x, y)(z) = \mu(x, y, z)$, be the adjoint representation of 3-Lie algebra L . Then $\text{ad}_\mu^* : L \wedge L \rightarrow \text{gl}(L^*)$, defined by

$$\langle \text{ad}_\mu^*(x, y)(\xi), z \rangle = -\langle \xi, \text{ad}_\mu(x, y)(z) \rangle = -\langle \xi, \mu(x, y, z) \rangle, \quad \forall x, y, z \in L, \xi \in L^*$$

is the coadjoint representation of L .

Theorem 3.3 Let (L, μ, Δ) be a 3-Lie bialgebra. Then (L^*, Δ^*, μ^*) is a 3-Lie bialgebra, and it is called the dual 3-Lie bialgebra of (L, μ, Δ) .

Proof Since (L, μ, Δ) is a 3-Lie bialgebra, by Theorem 2.3 and Theorem 2.4, (L^*, Δ^*) be a 3-Lie algebra in the multiplication (2.18), and (L^*, μ^*) be a 3-Lie coalgebra in the multiplication (2.6).

We will prove that $\mu^* : L^* \rightarrow L^* \wedge L^* \wedge L^*$ satisfies identity (3.1), that is, the following identity holds for every $\xi, \eta, \zeta \in L^*$

$$\mu^*(\Delta^*(\xi, \eta, \zeta)) = \text{ad}_{\Delta^*}^{(3)}(\xi, \eta) \mu^*(\zeta) + \text{ad}_{\Delta^*}^{(3)}(\eta, \zeta) \mu^*(\xi) + \text{ad}_{\Delta^*}^{(3)}(\zeta, \xi) \mu^*(\eta). \quad (3.3)$$

For every $x, y, z \in L$ and $\xi, \eta, \zeta \in L^*$, by identities (2.6) and (2.18)

$$\begin{aligned} \langle \mu^* \Delta^*(\xi, \eta, \zeta), x \otimes y \otimes z \rangle &= \\ \langle \Delta^*(\xi, \eta, \zeta), \mu(x, y, z) \rangle &= \langle \xi \otimes \eta \otimes \zeta, \Delta(\mu(x, y, z)) \rangle = \\ \langle \xi \otimes \eta \otimes \zeta, \text{ad}_\mu^{(3)}(x, y) \Delta(z) \rangle &+ \langle \xi \otimes \eta \otimes \zeta, \text{ad}_\mu^{(3)}(z, x) \Delta(y) \rangle + \langle \xi \otimes \eta \otimes \zeta, \text{ad}_\mu^{(3)}(y, z) \Delta(x) \rangle. \end{aligned}$$

Without loss of generality, suppose $\Delta(z) = \sum_i a_i \otimes b_i \otimes c_i$, where $a_i, b_i, c_i \in L$. Then

$$\begin{aligned}
\langle \xi \otimes \eta \otimes \zeta, ad_\mu^{(3)}(x, y)\Delta(z) \rangle &= \langle \xi \otimes \eta \otimes \zeta, ad_\mu^{(3)}(x, y)(\sum_i a_i \otimes b_i \otimes c_i) \rangle = \\
&= \langle \xi \otimes \eta \otimes \zeta, \sum_i (ad_\mu(x, y)(a_i) \otimes b_i \otimes c_i + a_i \otimes ad_\mu(x, y)(b_i) \otimes c_i + a_i \otimes b_i \otimes ad_\mu(x, y)(c_i)) \rangle \\
&= -\sum_i \langle ad_\mu^*(x, y)(\xi) \otimes \eta \otimes \zeta + \xi \otimes ad_\mu^*(x, y)(\eta) \otimes \zeta + \xi \otimes \eta \otimes ad_\mu^*(x, y)(\zeta), a_i \otimes b_i \otimes c_i \rangle \\
&= -\langle ad_\mu^*(x, y)(\xi) \otimes \eta \otimes \zeta + \xi \otimes ad_\mu^*(x, y)(\eta) \otimes \zeta + \xi \otimes \eta \otimes ad_\mu^*(x, y)(\zeta), \Delta(z) \rangle \\
&= -\langle \Delta^*(ad_\mu^*(x, y)(\xi), \eta, \zeta) + \Delta^*(\xi, ad_\mu^*(x, y)(\eta), \zeta) + \Delta^*(\xi, \eta, ad_\mu^*(x, y)(\zeta)), z \rangle \\
&= -\langle ad_{\Delta^*}(\eta, \zeta)(ad_\mu^*(x, y)(\xi) + ad_{\Delta^*}(\zeta, \xi)ad_\mu^*(x, y)(\eta) + ad_{\Delta^*}(\xi, \eta)ad_\mu^*(x, y)(\zeta)), z \rangle. \quad (3.4)
\end{aligned}$$

Similarly, we have

$$\begin{aligned}
&\langle \xi \otimes \eta \otimes \zeta, ad_\mu^{(3)}(z, x)\Delta(y) \rangle \\
&= -\langle ad_{\Delta^*}(\eta, \zeta)(ad_\mu^*(z, x)(\xi) + ad_{\Delta^*}(\zeta, \xi)ad_\mu^*(z, x)(\eta) + ad_{\Delta^*}(\xi, \eta)ad_\mu^*(z, x)(\zeta)), y \rangle, \quad (3.5) \\
&\langle \xi \otimes \eta \otimes \zeta, ad_\mu^{(3)}(y, z)\Delta(x) \rangle \\
&= -\langle ad_{\Delta^*}(\eta, \zeta)(ad_\mu^*(y, z)(\xi) + ad_{\Delta^*}(\zeta, \xi)ad_\mu^*(y, z)(\eta) + ad_{\Delta^*}(\xi, \eta)ad_\mu^*(y, z)(\zeta)), x \rangle. \quad (3.6)
\end{aligned}$$

Since

$$\begin{aligned}
&-\langle ad_{\Delta^*}(\eta, \zeta)ad_\mu^*(x, y)(\xi), z \rangle - \langle ad_{\Delta^*}(\eta, \zeta)ad_\mu^*(y, z)(\xi), x \rangle - \langle ad_{\Delta^*}(\eta, \zeta)ad_\mu^*(z, x)(\xi), y \rangle \\
&= \langle ad_\mu^*(x, y)(\xi), ad_{\Delta^*}^*(\eta, \zeta)z \rangle + \langle ad_\mu^*(y, z)(\xi), ad_{\Delta^*}^*(\eta, \zeta)x \rangle + \langle ad_\mu^*(z, x)(\xi), ad_{\Delta^*}^*(\eta, \zeta)y \rangle \\
&= -\langle \xi, \mu(x, y, ad_{\Delta^*}^*(\eta, \zeta)z) \rangle - \langle \xi, \mu(ad_{\Delta^*}^*(\eta, \zeta)x, y, z) \rangle - \langle \xi, \mu(x, ad_{\Delta^*}^*(\eta, \zeta)y, z) \rangle \\
&= -\langle \mu^*(\xi), ad_{\Delta^*}^{*(3)}(\eta, \zeta)(x \otimes y \otimes z) \rangle = \langle ad_{\Delta^*}^{(3)}(\eta, \zeta)\mu^*(\xi), x \otimes y \otimes z \rangle, \text{ and} \\
&-\langle ad_{\Delta^*}(\zeta, \xi)ad_\mu^*(x, y)(\eta), z \rangle - \langle ad_{\Delta^*}(\zeta, \xi)ad_\mu^*(y, z)(\eta), x \rangle - \langle ad_{\Delta^*}(\zeta, \xi)ad_\mu^*(z, x)(\eta), y \rangle \\
&= -\langle \mu^*(\eta), ad_{\Delta^*}^{*(3)}(\zeta, \xi)(x \otimes y \otimes z) \rangle = \langle ad_{\Delta^*}^{(3)}(\zeta, \xi)\mu^*(\eta), x \otimes y \otimes z \rangle, \text{ and} \\
&-\langle ad_{\Delta^*}(\xi, \eta)ad_\mu^*(x, y)(\zeta), z \rangle - \langle ad_{\Delta^*}(\xi, \eta)ad_\mu^*(y, z)(\zeta), x \rangle - \langle ad_{\Delta^*}(\xi, \eta)ad_\mu^*(z, x)(\zeta), y \rangle \\
&= -\langle \mu^*(\eta), ad_{\Delta^*}^{*(3)}(\zeta, \xi)(x \otimes y \otimes z) \rangle = \langle ad_{\Delta^*}^{(3)}(\zeta, \xi)\mu^*(\eta), x \otimes y \otimes z \rangle,
\end{aligned}$$

the identity (3.3) holds. It follows the result. \square

By the duality property, every 3-Lie bialgebra has dual 3-Lie bialgebra whose dual is the 3-Lie bialgebra itself. And summarizing identities (3.2) - (3.6), we obtain

$$\begin{aligned}
&\langle \Delta^*(\xi, \eta, \zeta), \mu(x, y, z) \rangle \\
&= -\langle ad_{\Delta^*}(\eta, \zeta)(ad_\mu^*(x, y)(\xi) + ad_{\Delta^*}(\zeta, \xi)ad_\mu^*(x, y)(\eta) + ad_{\Delta^*}(\xi, \eta)ad_\mu^*(x, y)(\zeta)), z \rangle - \\
&\langle ad_{\Delta^*}(\eta, \zeta)(ad_\mu^*(z, x)(\xi) + ad_{\Delta^*}(\zeta, \xi)ad_\mu^*(z, x)(\eta) + ad_{\Delta^*}(\xi, \eta)ad_\mu^*(z, x)(\zeta)), y \rangle - \\
&\langle ad_{\Delta^*}(\eta, \zeta)ad_\mu^*(y, z)(\xi) + ad_{\Delta^*}(\zeta, \xi)ad_\mu^*(y, z)(\eta) + ad_{\Delta^*}(\xi, \eta)ad_\mu^*(y, z)(\zeta)), x \rangle. \quad (3.7)
\end{aligned}$$

Example 3.4 From Theorem 3.3, the dual 3-Lie bialgebra (L^*, Δ^*, μ^*) of (L, μ, Δ) in Example 3.2 satisfying $\langle e^i, e_j \rangle = \delta_{ij}$, $1 \leq i, j \leq 4$, and

$$\mu^* : L^* \rightarrow L^* \wedge L^* \wedge L^*, \quad \begin{cases} \mu^*(e^1) = e^1 \wedge e^3 \wedge e^4, \\ \mu^*(e^2) = e^2 \wedge e^3 \wedge e^4; \end{cases}$$

$$\Delta^* : L^* \wedge L^* \wedge L^* \rightarrow L^*, \quad \begin{cases} \Delta^*(e^3 \wedge e^2 \wedge e^4) = e^1, \\ \Delta^*(e^1 \wedge e^3 \wedge e^4) = e^3. \end{cases} \quad \square$$

At last of this section we describe 3-Lie bialgebras by the structural constants.

Let (L, μ, Δ) be a 3-Lie bialgebra with the multiplications in the basis e_1, \dots, e_m as follows

$$\mu(e_i, e_j, e_k) = \sum_{l=1}^m c_{ijk}^l e_l, \quad \Delta(e_l) = \sum_{1 \leq i < j < k \leq m} a_l^{ijk} e_i \wedge e_j \wedge e_k, \quad (3.8)$$

where $c_{ijk}^l, a_l^{ijk} \in F$ $1 \leq i < j < k \leq m, 1 \leq l \leq m$, satisfy identities (2.14) and (2.15), respectively. Then

$$\Delta\mu(e_i, e_j, e_k) = \sum_{l=1}^m c_{ijk}^l \Delta(e_l) = \sum_{l=1}^m \sum_{r < s < t} c_{ijk}^l a_l^{rst} e_r \wedge e_s \wedge e_t, \quad 1 \leq i < j < k \leq m. \quad (3.9)$$

By identity (3.1),

$$\begin{aligned} \Delta\mu(e_i, e_j, e_k) &= ad_\mu^{(3)}(e_i, e_j)\Delta(e_k) + ad_\mu^{(3)}(e_j, e_k)\Delta(e_i) + ad_\mu^{(3)}(e_k, e_i)\Delta(e_j) \\ &= \sum_{r < s < t} a_k^{rst} [\mu(e_i, e_j, e_r) \wedge e_s \wedge e_t + e_r \wedge \mu(e_i, e_j, e_s) \wedge e_t + e_r \wedge e_s \wedge \mu(e_i, e_j, e_t)] \\ &+ \sum_{r < s < t} a_i^{rst} [\mu(e_j, e_k, e_r) \wedge e_s \wedge e_t + e_r \wedge \mu(e_j, e_k, e_s) \wedge e_t + e_r \wedge e_s \wedge \mu(e_j, e_k, e_t)] \\ &+ \sum_{r < s < t} a_j^{rst} [\mu(e_k, e_i, e_r) \wedge e_s \wedge e_t + e_r \wedge \mu(e_k, e_i, e_s) \wedge e_t + e_r \wedge e_s \wedge \mu(e_k, e_i, e_t)] \\ &= \sum_{r < s < t} \sum_{l=1}^m a_k^{rst} [c_{ijr}^l e_l \wedge e_s \wedge e_t + c_{ijs}^l e_r \wedge e_l \wedge e_t + c_{ijt}^l e_r \wedge e_s \wedge e_l] \\ &+ \sum_{r < s < t} \sum_{l=1}^m a_i^{rst} [c_{jkr}^l e_l \wedge e_s \wedge e_t + c_{jks}^l e_r \wedge e_l \wedge e_t + c_{jkt}^l e_r \wedge e_s \wedge e_l] \\ &+ \sum_{r < s < t} \sum_{l=1}^m a_j^{rst} [c_{kir}^l e_l \wedge e_s \wedge e_t + c_{kis}^l e_r \wedge e_l \wedge e_t + c_{kit}^l e_r \wedge e_s \wedge e_l]. \end{aligned} \quad (3.10)$$

Comparing the identities (3.9) and (3.10), we obtain

$$\begin{aligned} &a_k^{rst}(c_{ijr}^r + c_{ijs}^s + c_{ijt}^t) + a_i^{rst}(c_{jkr}^r + c_{jks}^s + c_{jkt}^t) + a_j^{rst}(c_{kir}^r + c_{kis}^s + c_{kit}^t) \\ &= \sum_{l=1}^m c_{ijk}^l a_l^{rst}, \quad 1 \leq i < j < k \leq m, \quad 1 \leq r < s < t \leq m; \end{aligned} \quad (3.11)$$

$$\sum_{l \neq r, s, t} [c_{ijr}^l a_k^{rst} + c_{kir}^l a_j^{rst} + c_{jkr}^l a_i^{rst}] = 0, \quad 1 \leq i < j < k \leq m, \quad 1 \leq r < s < t \leq m; \quad (3.12)$$

$$\sum_{l \neq r, s, t} [c_{ijs}^l a_k^{rst} + c_{kis}^l a_j^{rst} + c_{jks}^l a_i^{rst}] = 0, \quad 1 \leq i < j < k \leq m, \quad 1 \leq r < s < t \leq m; \quad (3.13)$$

$$\sum_{l \neq r, s, t} [c_{ijt}^l a_k^{rst} + c_{kit}^l a_j^{rst} + c_{jkt}^l a_i^{rst}] = 0, \quad 1 \leq i < j < k \leq m, \quad 1 \leq r < s < t \leq m. \quad (3.14)$$

Conversely, if a 3-Lie algebra (L, μ) and a 3-Lie coalgebra (L, Δ) defined by (3.8), respectively, satisfy identities (3.11)-(3.14), then μ, Δ satisfy identity (3.1). This proves the following result.

Theorem 3.5 Let L be a vector space with a basis e_1, \dots, e_m , (L, μ) and (L, Δ) be 3-Lie

algebra and 3-Lie coalgebra defined by (3.8). Then (L, μ, Δ) is a 3-Lie bialgebra if and only if c_{ijk}^l and a_l^{rst} , $1 \leq i < j < k \leq m, 1 \leq l \leq m$, satisfy identities (3.11)-(3.14). \square

Definition 3.6 Two 3-Lie bialgebras (L_1, μ_1, Δ_1) and (L_2, μ_2, Δ_2) are called equivalent if there exists a vector space isomorphism $f : L_1 \rightarrow L_2$ such that

(1) $f : (L_1, \mu_1) \rightarrow (L_2, \mu_2)$ is a 3-Lie algebra isomorphism, that is,

$$f\mu_1(x, y, z) = \mu_2(f(x), f(y), f(z)) \text{ for } \forall x, y, z \in L_1;$$

(2) $f : (L_1, \Delta_1) \rightarrow (L_2, \Delta_2)$ is a 3-Lie coalgebra isomorphism, that is,

$$\Delta_2(f(x)) = (f \otimes f \otimes f)\Delta_1(x) \text{ for every } x \in L_1. \square$$

For a given 3-Lie algebra L , in order to find all the 3-Lie bialgebra structures on L , we should find all the 3-Lie coalgebra structures on L that are compatible with the 3-Lie algebra L . Although a permutation of the basis of L gives the equivalent 3-Lie coalgebra structure on L , it may leads to a different 3-Lie bialgebra structure on L .

Example 3.7 Let L be a 4-dimensional 3-Lie algebra with a basis e_1, e_2, e_3, e_4 , and the multiplication $\mu : L \otimes L \otimes L \rightarrow L$ as $\begin{cases} \mu(e_2, e_3, e_4) = e_1, \\ \mu(e_1, e_3, e_4) = e_2. \end{cases}$ Defining the three linear mappings $\Delta_1, \Delta_2, \Delta_3 : L \rightarrow L \otimes L \otimes L$ as follows

$$\begin{cases} \Delta_1 e_1 = e_1 \wedge e_2 \wedge e_4, \\ \Delta_1 e_3 = e_3 \wedge e_2 \wedge e_4, \\ \Delta_1 e_2 = \Delta_1 e_4 = 0; \end{cases} \quad \begin{cases} \Delta_2 e_1 = e_1 \wedge e_4 \wedge e_2, \\ \Delta_2 e_3 = e_3 \wedge e_4 \wedge e_2, \\ \Delta_2 e_2 = \Delta_2 e_4 = 0; \end{cases} \quad \begin{cases} \Delta_3 e_2 = e_2 \wedge e_3 \wedge e_1, \\ \Delta_3 e_4 = e_4 \wedge e_3 \wedge e_1, \\ \Delta_3 e_1 = \Delta_3 e_3 = 0; \end{cases}$$

we obtain three isomorphic 3-Lie coalgebras $(L, \Delta_1), (L, \Delta_2), (L, \Delta_3)$.

By the direct computation, $(L, \mu, \Delta_1), (L, \mu, \Delta_2)$ and (L, μ, Δ_3) are 3-Lie bialgebras. Let $f : L \rightarrow L$ be a linear isomorphism defined by:

$$f(e_1) = e_2, \quad f(e_2) = -e_1, \quad f(e_3) = e_4, \quad f(e_4) = e_3.$$

Then (L, μ, Δ_1) and (L, μ, Δ_3) are equivalent 3-Lie bialgebras in the isomorphism $f : (L, \mu, \Delta_1) \rightarrow (L, \mu, \Delta_3)$. But (L, μ, Δ_2) is not equivalent to (L, μ, Δ_1) . \square

4. Classification of 3-dimensional 3-Lie bialgebras

In this section we classify the 3-dimensional 3-Lie bialgebras over an algebraically closed field F of characteristic zero.

Lemma 4.1^[14] Let (L, μ) be an m -dimensional 3-Lie algebra with a basis e_1, \dots, e_m , $m \leq 4$. Then up to isomorphisms there is one and only one of the following possibilities:

(1) $\dim L \leq 2$, L is abelian, that is, $\mu = 0$.

(2) $\dim L = 3$, L_a is abelian; $L_b : \mu_b(e_1, e_2, e_3) = e_1$.

(3) $\dim L = 4$, $L_a : L$ is abelian; $L_{b_1} : \mu_{b_1}(e_2, e_3, e_4) = e_1$; $L_{b_2} : \mu_{b_2}(e_1, e_2, e_3) = e_1$;

$$L_{c_1} \begin{cases} \mu_{c_1}(e_2, e_3, e_4) = e_1, \\ \mu_{c_1}(e_1, e_3, e_4) = e_2; \end{cases} \quad L_{c_2} \begin{cases} \mu_{c_2}(e_2, e_3, e_4) = \alpha e_1 + e_2, \\ \mu_{c_2}(e_1, e_3, e_4) = e_2, \quad \alpha \in F, \alpha \neq 0; \end{cases}$$

$$L_{c_3} \begin{cases} \mu_{c_3}(e_1, e_3, e_4) = e_1, \\ \mu_{c_3}(e_2, e_3, e_4) = e_2; \end{cases} \quad L_d \begin{cases} \mu_d(e_2, e_3, e_4) = e_1, \\ \mu_d(e_1, e_3, e_4) = e_2, \\ \mu_d(e_1, e_2, e_4) = e_3; \end{cases} \quad L_e \begin{cases} \mu_e(e_2, e_3, e_4) = e_1, \\ \mu_e(e_1, e_3, e_4) = e_2, \\ \mu_e(e_1, e_2, e_4) = e_3, \\ \mu_e(e_1, e_2, e_3) = e_4. \end{cases} \quad \square$$

Following Theorem 2.3, Theorem 2.4 and Lemma 4.1, we have the classification of m -dimensional 3-Lie coalgebras with $m \leq 4$.

Lemma 4.2 Let (L, Δ) be an m -dimensional 3-Lie coalgebra with a basis e^1, \dots, e^m , $m \leq 4$. Then up to isomorphisms there is one and only one of the following possibilities:

- (1) $\dim L \leq 2$, (L, Δ_a) is trivial, that is, $\Delta_a = 0$.
- (2) $\dim L = 3$, L is trivial with $\Delta = 0$; C_b : $\Delta_b(e^1) = e^1 \wedge e^2 \wedge e^3$.
- (3) $\dim L = 4$, C_a : (L, Δ_a) is trivial;

$$C_{b_1} : \Delta_{b_1}(e^1) = e^2 \wedge e^3 \wedge e^4; \quad C_{b_2} : \Delta_{b_2}(e^1) = e^1 \wedge e^2 \wedge e^3;$$

$$C_{c_1} \begin{cases} \Delta_{c_1}(e^1) = e^2 \wedge e^3 \wedge e^4, \\ \Delta_{c_1}(e^2) = e^1 \wedge e^3 \wedge e^4; \end{cases} \quad C_{c_2} \begin{cases} \Delta_{c_2}(e^1) = \alpha e^2 \wedge e^3 \wedge e^4, \alpha \in F, \alpha \neq 0, \\ \Delta_{c_2}(e^2) = e^2 \wedge e^3 \wedge e^4 + e^1 \wedge e^3 \wedge e^4; \end{cases}$$

$$C_{c_3} \begin{cases} \Delta_{c_3}(e^1) = e^1 \wedge e^3 \wedge e^4, \\ \Delta_{c_3}(e^2) = e^2 \wedge e^3 \wedge e^4; \end{cases} \quad C_d \begin{cases} \Delta_d(e^1) = e^2 \wedge e^3 \wedge e^4, \\ \Delta_d(e^2) = e^1 \wedge e^3 \wedge e^4, \\ \Delta_d(e^3) = e^1 \wedge e^2 \wedge e^4; \end{cases} \quad C_e \begin{cases} \Delta_e(e^1) = e^2 \wedge e^3 \wedge e^4, \\ \Delta_e(e^2) = e^1 \wedge e^3 \wedge e^4, \\ \Delta_e(e^3) = e^1 \wedge e^2 \wedge e^4, \\ \Delta_e(e^4) = e^1 \wedge e^2 \wedge e^3. \end{cases}$$

Proof The results follow from Theorem 2.3, Theorem 2.5 and Lemma 3.1, directly. \square

Theorem 4.3 Let (L, μ, Δ) be a 3-dimensional 3-Lie bialgebra, then L is equivalent to one and only one of the following possibilities:

- $(L, 0, 0)$, that is, $\mu = 0$ and $\Delta = 0$;
- $(L, 0, \Delta_1)$: $\mu = 0, \Delta_1 e_1 = e_1 \wedge e_2 \wedge e_3$;
- $(L, \mu_b, 0)$: $\Delta = 0$;
- (L, μ_b, Δ_1) : $\Delta_1 e_1 = e_1 \wedge e_2 \wedge e_3$;
- (L, μ_b, Δ_2) : $\Delta_2(e_2) = e_2 \wedge e_3 \wedge e_1$;
- (L, μ_b, Δ_3) : $\Delta_3(e_1) = e_1 \wedge e_3 \wedge e_2$,

where μ_b is defined in Lemma 4.1.

Proof If (L, μ) is the abelian 3-Lie algebra, then by Lemma 4.1 and Lemma 4.2, non-equivalent 3-Lie bialgebras are only $(L, 0, 0)$ and $(L, 0, \Delta_1)$.

If (L, μ) is a 3-Lie algebra with $\mu = \mu_b$, then by the direct computation, the possibilities of 3-Lie bialgebras (L, μ_b, Δ_i) are as follows: $(L, \mu_b, 0)$;

$$(L, \mu_b, \Delta_1): \Delta_1(e_1) = e_1 \wedge e_2 \wedge e_3, \Delta_1(e_2) = \Delta_1(e_3) = 0;$$

$$(L, \mu_b, \Delta_2): \Delta_2(e_2) = e_2 \wedge e_3 \wedge e_1, \Delta_2(e_1) = \Delta_2(e_3) = 0;$$

$$\begin{aligned}
(L, \mu_b, \Delta_3): \Delta_3(e_1) &= e_1 \wedge e_3 \wedge e_2, \Delta_3(e_2) = \Delta_3 e_3 = 0; \\
(L, \mu_b, \Delta_4): \Delta_4(e_2) &= e_2 \wedge e_1 \wedge e_3, \Delta_4(e_1) = \Delta_4 e_3 = 0; \\
(L, \mu_b, \Delta_5): \Delta_5(e_3) &= e_3 \wedge e_1 \wedge e_2, \Delta_5(e_1) = \Delta_5 e_2 = 0; \\
(L, \mu_b, \Delta_6): \Delta_6(e_3) &= e_3 \wedge e_2 \wedge e_1, \Delta_6(e_1) = \Delta_6 e_2 = 0.
\end{aligned}$$

Defines linear mappings $f_i : L \rightarrow L$ for $i = 1, 2$ as follows:

$$f_1(e_1) = e_1, f_1(e_2) = e_3, f_1(e_3) = -e_2; \quad f_2(e_1) = e_1, f_2(e_2) = -e_2, f_2(e_3) = -e_3.$$

Then by the direct computation (L, μ_b, Δ_2) is equivalent to (L, μ_b, Δ_5) , (L, μ_b, Δ_4) is equivalent to (L, μ_b, Δ_6) in the mapping f_1 , respectively. (L, μ_b, Δ_2) is equivalent to (L, μ_b, Δ_4) in the mapping f_2 . And (L, μ_b, Δ_i) is not equivalent to (L, μ_b, Δ_j) for $1 \leq i \neq j \leq 3$. \square

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